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Decay rate of solutions of Navier–Stokes equations in 3-dimensional half space

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Abstract

In this article, we derive the L^2 spatial decay rate of a weak solution of the incompressible flow in the half space. When the half space (or some other fluid region with boundary) is concerned, pressure estimate is main obstacle since we do not have enough information of the pressure on the boundary. In this paper, we give an idea which does not require any pressure information on the boundary, and we hope that our idea could be applied to other problems such as exterior domain problem.

Keywords: Navier–Stokes; Spatial decay rate; Half space; Weak solution

1. Introduction

Let $\mathbf{u}$ and p be the velocity and pressure, respectively, of the incompressible fluid in a half space. Let us consider Navier–Stokes equations

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} + \nabla p = 0, \\ &\operatorname{div}\mathbf{u} = 0 \quad \text{in } (x,t) \in \mathbb{R}^3_+ \times (0,\infty) \end{aligned} \tag{1}$$

with no slip boundary condition $\mathbf{u}(x_1, x_2, 0, t) = 0$ and initial data $\mathbf{u}(x, 0) = \mathbf{u}_0(x)$.

A weak solutions in global time have been constructed by several mathematicians. (See [5,9,12,13,17–19], etc.) The main idea was highlighted on Galerkin approximation to get energy inequality. The uniqueness or the existence of strong solution has been known only for small data. Concerning the uniqueness and the existence of strong solution, we are led to verifying the regularity of weak solutions. But, for the different questions like the stability and asymptotic

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analysis, we are interested in the estimate of the decay rates of the weak and strong solution. For the both aims, the idea of a weak solution satisfying "generalized energy inequality" in the sense of Cafarelli, Kohn and Nirenberg is useful. In [5] a weak solution satisfying several properties such as generalized energy inequality and measurable pressure is introduced, is called suitably weak solution, and constructed for a bounded domain and $\mathbb{R}^3$. (A similar concept such as suitable weak solution was introduced in [9] for an exterior domain, and [18] for any domain with boundary.) Weak solution satisfying "generalized energy inequality" allows us to apply various test functions since the pressure is now a measurable function and we can obtain weighted norm estimates from variational formulation.

When regularity estimate or decay estimate are concerned in the whole space $\mathbb{R}^3$, the pressure representation in terms of velocity is useful. From the pressure representation, one could say the effect of pressure p is almost the same as the square of velocity $|\mathbf{u}|^2$.

The situation is not simple when we consider a domain with boundary. For example, if we try to have energy estimate, we might meet the following integral:

$$\int_{\Omega} (\phi^2\mathbf{u}) \cdot \nabla p\, dx = -\int_{\Omega} p(\mathbf{u} \cdot \nabla\phi^2)\, dx,$$

where ϕ may be a cut-off function for the localization, or a weight such as $(1 + |x|^2)$ for the spatial decay estimate.

As it is seen in the above identity, the pressure causes a trouble. Unfortunately, the pressure has non-local property and we have not enough information of the pressure on the boundary. This fact makes it difficult to derive norm estimate when the boundary is involved with.

In this paper, we intend to derive a spatial decay estimate (uniform in time) of the weak solution of the Navier–Stokes equations. For this purpose we suggest an idea treating energy estimate for the half space as fluid region, and we can avoid the computations involved with the pressure. Suppose ϕ is a weight function vanishing near boundary of the half space. Introducing a test function $\operatorname{curl}(\phi\mathbf{v})$, we have an identity

$$\int_{\Omega} \operatorname{curl}(\phi\mathbf{v}) \cdot \nabla p\, dx = -\int_{\Omega} p \operatorname{div}\operatorname{curl}(\phi\mathbf{v})\, dx = 0. \tag{2}$$

The above observation will be useful in this paper to treat energy estimate with spatial weight. Here, we set

$$\mathbf{v} = \int_{\mathbb{R}^3_+} N(x-y)[\phi \operatorname{curl}\mathbf{u}](y)\, dy,$$

where $N(y) = \frac{1}{4\pi}\frac{1}{|y|}$, $\phi \in C_0^\infty(\mathbb{R}^3_+)$ with $\phi(x) = O(x_3^2)$.

In Section 2 we see that the auxiliary vector filed $\mathbf{v}$ is rather good alternative of $\phi\mathbf{u}$, for example, we see that

$$-\Delta\mathbf{v} = \phi\nabla \times \mathbf{u}.$$

In Section 3, we consider a weak solution of (1). We show that $(1+t)^{5/8}\|(1+x_3^2)^{1/2}\mathbf{u}(t)\|_{L^2}$ is bounded uniform in t. For this purpose we consider an approximating solution $\mathbf{u}^n$ and an auxiliary vector filed $\mathbf{v}_R^n$ defined by $\mathbf{v}_R^n(x) = \int_{\mathbb{R}^3_+} N(x-y)\phi_R(y)\operatorname{curl}_y \mathbf{u}^n(y)\, dy$, $\phi_R(x) = x_3^2\zeta_R^2(x)$, where ζ_R is a cut-off function supported in B_{2R}. By energy estimate after removing the pressure by the special form of test function, in Section 3.1 we show that $\|\nabla\mathbf{v}_R^n(t)\|_{L^2}$ is uniformly bounded in t, R and n, and then we show that the uniform boundedness of $\|\nabla\mathbf{v}_R^n(t)\|_{L^2}$ implies the uniform boundedness of $(1+t)^{5/8}\|\phi_R^{1/2}\mathbf{u}^n(t)\|_2$. By sending R to the infinity and by lower semi-continuity, we will obtain the uniform estimate of $(1+t)^{5/8}\|x_3\mathbf{u}(t)\|_2$.

There are abundant literature on the temporal decay rate of solutions of Navier–Stokes equations by many mathematicians. (See [1–3,7–9,14–16,20–23,25] and references therein.) Spatial decay estimate have been considered by Brandolese [4], Farwig and Sohr [6], He [10], He and Xin [11], Takahashi [24], etc.

When the whole space $\mathbb{R}^3$ is concerned, we have a uniform bound in L^2 space with a spatial weight. While a fluid region with boundary is considered, the situation is no more simple. Since we do not have pressure representation in arbitrary domain, it is difficult to make use of energy method. In [11], He and Xin considered exterior domain problem. By removing pressure in solution representation, they obtained a decay rate estimate in space and time for the strong solution for a small data.

We are indebted to the paper of Z. Xin and C. He [11] in the sense that the main difficulties have been overcome by removing pressure term.

We state our main theorem whose proof is the main purpose of the subsequent sections.

Theorem 1.1. *Let* $\mathbf{u}_0 \in L^1 \cap L^2$, $x_3\mathbf{u}_0 \in L^1 \cap L^{6/5}$, $x_3^2\mathbf{u}_0 \in L^2$, $\operatorname{div}\mathbf{u}_0 = 0$. *Then there is a weak solution* $\mathbf{u}$ *of the Navier–Stokes equation* (1) *in the half space satisfying that*

$$\left\|x_3\mathbf{u}(t)\right\|_{L^2(\mathbb{R}^3_+)} \leqslant C(1+t)^{-\frac{5}{8}}$$

for some C independent of t.

As far as half space is concerned, the temporal decay rate of the solution ($1 < p \leqslant 2$ for weak solution, $2 < p \leqslant \infty$ for strong solution) of the Navier–Stokes equation is known as follows (see [8]):

$$\left\|\mathbf{u}(t)\right\|_{L^p(\mathbb{R}^3_+)} \leqslant c(1+t)^{-2+\frac{3}{2p}} \quad \text{for } 1 < p \leqslant \infty. \tag{3}$$

Throughout this paper, we assume the above estimate (3) as preliminary one and the proof of our main theorem is based on the above estimate.

2. Preliminary estimates

Suppose $\mathbf{u}$ is a regular enough vector field in the half space. Let N be the fundamental function of $-\Delta$, that is, $N = N(x-y) = \frac{1}{4\pi}\frac{1}{|x-y|}$. Define ϕ be a smooth function which vanishes at $x_3 = 0$ and near $x = \infty$. We set

$$\mathbf{v} = \int_{\mathbb{R}^3_+} N(x-y)\big[\phi(y)(\operatorname{curl}\mathbf{u})(y)\big]\,dy.$$

Then $\mathbf{v}$ is defined in the whole space $\mathbb{R}^3$. By definition of $\mathbf{v}$ we have

$$-\Delta\mathbf{v} = \begin{cases} \phi\operatorname{curl}\mathbf{u}, & x_3 > 0, \\ 0, & x_3 < 0. \end{cases}$$

We observe that if ϕ is compactly supported, then $\mathbf{v} \to 0$ as $|x| \to \infty$. Moreover, we have

$$\operatorname{curl}\mathbf{v} = \int_{\mathbb{R}^3_+} N(x-y)\operatorname{curl}\big[\phi(\operatorname{curl}\mathbf{u})\big](y)\,dy = \phi\mathbf{u} + \mathbf{A}_1,$$

where

$$\mathbf{A}_1 = \nabla\int_{\mathbb{R}^3_+} N(x-y)\big[(\mathbf{u}\cdot\nabla)\phi\big](y)\,dy - \operatorname{curl}\int_{\mathbb{R}^3_+} N(x-y)[\nabla\phi\times\mathbf{u}](y)\,dy.$$

The above estimates comes from the following observations:

$$\begin{aligned} \operatorname{curl}\big[\phi(y)(\operatorname{curl}\mathbf{u})(y)\big] &= \operatorname{curl}\big[\operatorname{curl}(\phi\mathbf{u})\big] - \operatorname{curl}\big[(\nabla\phi)\times\mathbf{u}\big] \\ &= -\Delta(\phi\mathbf{u}) + \nabla\big[(\mathbf{u}\cdot\nabla)\phi\big] - \operatorname{curl}\big[(\nabla\phi)\times\mathbf{u}\big]. \end{aligned}$$

In this and the next section, we define $\phi_R(x) = x_3^2\zeta_R^2(x)$, where

$$\zeta_R \in C_0^\infty\big(\mathbb{R}^3\big) \quad \text{with} \quad \zeta_R(x) = 1 \quad \text{on } |x| \leqslant R, \quad \zeta_R(x) = 0 \quad \text{on } |x| \geqslant 2R.$$

We observe that $|\nabla\phi_R| \leqslant c\phi_R^{1/2}$, $|\nabla^2\phi_R| \leqslant c$, and $\nabla^3\phi_R$ is compactly supported in $D_R = \{x\colon\ |x| \leqslant 2R\}$ with $|\nabla^3\phi_R| \leqslant \frac{c}{R}$. Here c is independent of R.

We set

$$\mathbf{v}_R = \int_{\mathbb{R}^3_+} N(x-y)\big[\phi_R(y)(\operatorname{curl}\mathbf{u})(y)\big]\,dy,$$

and

$$\mathbf{A}_{1,R} = \nabla \int\limits_{\mathbb{R}^3_+} N(x-y)\big[(\mathbf{u}\cdot\nabla)\phi_R\big](y)\,dy - \operatorname{curl}\int\limits_{\mathbb{R}^3_+} N(x-y)[\nabla\phi_R \times \mathbf{u}](y)\,dy.$$

By Sobolev imbedding theorem and Calderon–Zygmund inequality we have the following estimates for $\mathbf{v}_R$ and $\mathbf{A}_{1,R}$.

Lemma 2.1. *There is c independent of R and $\mathbf{u}$ that*

$$\|\mathbf{v}_R\|_{L^{3p/(3-2p)}} \leqslant c\|\nabla\mathbf{v}_R\|_{L^{3p/(3-p)}} \leqslant c\big\|\nabla^2\mathbf{v}_R\big\|_p \leqslant c\|\phi_R\nabla\mathbf{u}\|_p, \quad p\in(1,3/2).$$

Lemma 2.2. *There is c independent of R and $\mathbf{u}$ that*

$$\|\mathbf{A}_{1,R}\|_{L^{3p/(3-p)}} \leqslant c\|\nabla\mathbf{A}_{1,R}\|_p \leqslant c\big\|\phi_R^{\frac12}\mathbf{u}\big\|_p, \quad p\in(1,3).$$

The following lemma will be useful in Section 3.1.

Lemma 2.3. *There is c independent of R and $\mathbf{u}$ that*

$$\big\|\phi_R^{\frac12}\mathbf{u}\big\|_2^2 \leqslant c\|\phi_R\nabla\mathbf{u}\|_2\|\mathbf{u}\|_{\frac65} + c\|\mathbf{u}\|_{\frac65}^2.$$

Proof. By Young's inequality we have

$$\begin{aligned}
\big\|\phi_R^{\frac12}\mathbf{u}\big\|_2^2 &\leqslant \|\phi_R\mathbf{u}\|_6\|\mathbf{u}\|_{\frac65} \leqslant c\big\|\nabla(\phi_R\mathbf{u})\big\|_2\|\mathbf{u}\|_{\frac65}\\
&\leqslant c\|\phi_R\nabla\mathbf{u}\|_2\|\mathbf{u}\|_{\frac65} + c\big\|\phi_R^{\frac12}\mathbf{u}\big\|_2\|\mathbf{u}\|_{\frac65}\\
&\leqslant \epsilon\big\|\phi_R^{\frac12}\mathbf{u}\big\|_2^2 + c\|\phi_R\nabla\mathbf{u}\|_2\|\mathbf{u}\|_{\frac65} + c_\epsilon\|\mathbf{u}\|_{\frac65}^2
\end{aligned}$$

for some constant C_ϵ depending on ϵ. Now take $\epsilon < \frac12$. □

3. Proof of Theorem 1.1

In this section we consider the decay rate of weak solution $\mathbf{u}$ in $L^2(\mathbb{R}^3_+)$ with spatial weight x_3. For this purpose, we consider approximating solution $\mathbf{u}^n$, which is introduced in Appendix A and then in Section 3.1 we derive decay rate as follows:

$$(1+t)^{\frac58}\big\|x_3\mathbf{u}^n(t)\big\|_{L^2(\mathbb{R}^3_+)} \leqslant C \quad \text{uniform in } t, n.$$

Considering the weak limit of $\mathbf{u}$ as n goes to the infinity and by lower semi-continuity, we obtain

$$(1+t)^{\frac58}\big\|x_3\mathbf{u}(t)\big\|_{L^2(\mathbb{R}^3_+)} \leqslant C \quad \text{uniform in } t.$$

We use the notation $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{\alpha,p} = \|x_3^\alpha\cdot\|_p$ for short.

3.1. Spatial decay estimates of approximating solution

Let us consider the solution $\mathbf{u}$, p of the approximating equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = 0,\\
&\nabla\cdot\mathbf{u} = 0, \quad \text{in } \mathbb{R}^3_+\times(0,\infty),
\end{aligned} \tag{4}$$

with initial condition $\mathbf{u}(0) = \mathbf{u}_0$ in $\mathbb{R}^3_+$, and boundary condition $\mathbf{u} = 0$ on $x_3 = 0$ and $\mathbf{u} = 0$ at infinity. Here $\mathbf{U}$ is a spatial mollification of $\mathbf{u}$ with the property that

$$\left\|\nabla^k \mathbf{U}(t)\right\|_p \leqslant \left\|\nabla^k \mathbf{u}(t)\right\|_p, \quad 1 \leqslant p \leqslant \infty,\ k = 0, 1.$$

In this section we show the spatial decay estimates of a solution $\mathbf{u}$ of approximating equation (4) when the initial data satisfies $\mathbf{u}_0 \in L^1 \cap L^2$, $\operatorname{div} \mathbf{u}_0 = 0$.

Theorem 3.1. *Suppose that $x_3^2 \mathbf{u}_0 \in L^2$, $x_3 \mathbf{u}_0 \in L^{6/5}$, $\mathbf{u}_0 \in L^1 \cap L^2$, $\operatorname{div} \mathbf{u}_0 = 0$. Then there is C independent of $\mathbf{u}$ so that*

$$\left\|x_3 \mathbf{u}(t)\right\|_{L^2(\mathbb{R}^3_+)} \leqslant C(1+t)^{-\frac{5}{8}}.$$

For this purpose, we take an auxiliary vector filed $\mathbf{v}_R$ defined by

$$\mathbf{v}_R(x) = \int_{\mathbb{R}^3_+} N(x-y)\phi_R(y) \operatorname{curl}_y \mathbf{u}(y)\, dy, \qquad \phi_R(x) = x_3^2 \zeta_R^2(x),$$

where $\zeta_R(x) \in C_0^\infty(\mathbb{R}^3)$ is a cut-off function satisfying

$$0 \leqslant \zeta_R \leqslant 1, \quad \zeta_R(x) = 1 \quad \text{on } |x| \leqslant R, \quad \zeta_R(x) = 0 \quad \text{on } |x| \geqslant 2R.$$

The rest of this section will be devoted to prove the above theorem for approximating solution $\mathbf{u}$, and during this proof, we also derive estimate of $\|\nabla \mathbf{v}_R(t)\|_2$ (uniform in t, R and $\mathbf{u}$).

Multiply $\operatorname{curl}(\phi_R \mathbf{v}_R)$ to the both sides of $(4)_1$. Since

$$\int_{\mathbb{R}^3_+} \nabla p \cdot \left[\operatorname{curl}(\phi_R \mathbf{v}_R)\right] dx = 0,$$

we have the identity

$$\int_{\mathbb{R}^3_+} \frac{\partial}{\partial t}\mathbf{u} \cdot \left[\operatorname{curl}(\phi_R \mathbf{v}_R)\right] dx + \int_{\mathbb{R}^3_+} \left[(\mathbf{U}\cdot\nabla)\mathbf{u} - \Delta \mathbf{u}\right] \cdot \operatorname{curl}(\phi_R \mathbf{v}_R)\, dx = 0.$$

By integrations by parts, we observe that

$$\int_{\mathbb{R}^3_+} \frac{\partial}{\partial t}\mathbf{u} \cdot \operatorname{curl}(\phi_R \mathbf{v}_R)\, dx = \frac{d}{dt}\int_{\mathbb{R}^3_+} \mathbf{u} \cdot \operatorname{curl}(\phi_R \mathbf{v}_R)\, dx - \int_{\mathbb{R}^3_+} \mathbf{u} \cdot \operatorname{curl}(\phi_R \partial_t \mathbf{v}_R)\, dx.$$

We observe that

$$\int_{\mathbb{R}^3_+} \mathbf{u} \cdot \operatorname{curl}(\phi_R \partial_t \mathbf{v}_R)\, dx = \int_{\mathbb{R}^3_+} \operatorname{curl}(\phi_R \mathbf{v}_R) \cdot \big(\Delta \mathbf{u} - (\mathbf{U}\cdot\nabla)\mathbf{u}\big)\, dx,$$

since

$$\begin{aligned}
\partial_t \mathbf{v}_R(x) &= \int N(x-y)\phi_R(y) \operatorname{curl}(\partial_t \mathbf{u})(y)\, dy \\
&= \int N(x-y)\phi_R(y) \operatorname{curl}\big(\Delta \mathbf{u} - (\mathbf{U}\cdot\nabla)\mathbf{u}\big)(y)\, dy.
\end{aligned}$$

Hence, we have the identity

$$\frac{d}{dt}\int_{\mathbb{R}^3_+} \mathbf{u} \cdot \operatorname{curl}(\phi_R \mathbf{v}_R)\, dx + 2\int_{\mathbb{R}^3_+} \left[(\mathbf{U}\cdot\nabla)\mathbf{u} - \Delta \mathbf{u}\right] \cdot \operatorname{curl}(\phi_R \mathbf{v}_R)\, dx = 0. \tag{5}$$

Define $X_R(t)$ and $Y_R(t)$ by

$$X_R(t)=\int_{\mathbb{R}^3_+}\mathbf{u}\cdot\operatorname{curl}(\phi_R\mathbf{v}_R)\,dx,\qquad Y_R(t)=\int_{\mathbb{R}^3_+}\phi_R^2|\nabla\mathbf{u}|^2\,dx.$$

Observe that $\mathbf{v}_R$ is defined in the whole space $\mathbf{R}^3$ and thus

$$X_R(t)=\|\nabla\mathbf{v}_R\|^2_{L^2(\mathbb{R}^3)},$$

since

$$X_R(t)=\int_{\mathbb{R}^3}\mathbf{u}\cdot\operatorname{curl}(\phi_R\mathbf{v}_R)\,dx=\int_{\mathbb{R}^3}(\phi_R\operatorname{curl}\mathbf{u})\cdot\mathbf{v}_R\,dx=-\int_{\mathbb{R}^3}(\Delta\mathbf{v}_R)\cdot\mathbf{v}_R\,dx.$$

Since

$$\begin{aligned}
&\operatorname{curl}(\phi_R\mathbf{v}_R)=\phi_R^2\mathbf{u}+\phi_R\mathbf{A}_{1,R}+\nabla\phi_R\times\mathbf{v}_R,\\
&-\Delta\mathbf{u}\cdot\operatorname{curl}(\phi_R\mathbf{v}_R)=-\partial_i\left((\partial_i\mathbf{u})\phi_R^2-\frac{|\mathbf{u}|^2}{2}\partial_i\phi_R^2+(\partial_i\mathbf{u})(\phi_R\mathbf{A}_{1,R}+\nabla\phi_R\times\mathbf{v}_R)\right)\\
&\qquad\qquad+|\nabla\mathbf{u}|^2\phi_R^2-\frac{|\mathbf{u}|^2}{2}\Delta\phi_R^2+(\partial_i\mathbf{u})\cdot\partial_i(\phi_R A_{1,R}+\nabla\phi_R\times\mathbf{v}_R)
\end{aligned}$$

and

$$\begin{aligned}
\big[(\mathbf{U}\cdot\nabla)\mathbf{u}\big]\cdot\operatorname{curl}(\phi_R\mathbf{v}_R)&=\partial_i\left(U_i\left(\frac{|\mathbf{u}|^2}{2}\phi_R^2+\mathbf{u}\cdot(\phi_R A_{1,R}+\nabla\phi_R\times\mathbf{v}_R)\right)\right)\\
&\quad-\frac{|\mathbf{u}|^2}{2}(\mathbf{U}\cdot\nabla)\phi_R^2-\mathbf{u}\cdot(\mathbf{U}\cdot\nabla)\big[\phi_R\mathbf{A}_1+(\nabla\phi_R)\times\mathbf{v}_R\big].
\end{aligned}$$

Hence (5) reduces to the following identity:

$$\begin{aligned}
\frac{d}{dt}X_R(t)+2Y_R(t)&=\int_{\mathbb{R}^3_+}|\mathbf{u}|^2\Delta\phi_R^2\,dx+\int_{\mathbb{R}^3_+}|\mathbf{u}|^2(\mathbf{U}\cdot\nabla)\phi_R^2\,dx-2\int_{\mathbb{R}^3_+}(\partial_i\mathbf{u})\cdot\partial_i(\phi_R A_{1,R}+\nabla\phi_R\times\mathbf{v}_R)\,dx\\
&\quad+2\int_{\mathbb{R}^3_+}\mathbf{u}\cdot(\mathbf{U}\cdot\nabla)\big[\phi_R\mathbf{A}_{1,R}+(\nabla\phi_R)\times\mathbf{v}_R\big]\,dx\\
&=I+II+III+IV.
\end{aligned}$$

Note that

$$\begin{aligned}
III&=2\int_{\mathbb{R}^3_+}(\partial_i\mathbf{u})\cdot\big[(\phi_R\partial_i A_{1,R}+\partial_i\phi_R A_{1,R})+\nabla\phi_R\times\partial_i\mathbf{v}_R+(\nabla\partial_i\phi_R)\times\mathbf{v}_R\big]\,dx\\
&=2\int_{\mathbb{R}^3_+}(\partial_i\mathbf{u})\cdot\phi_R\partial_i A_{1,R}\,dx-2\int_{\mathbb{R}^3_+}\mathbf{u}\cdot\big[(\partial_i\phi_R)(\partial_i A_{1,R})+(\Delta\phi_R)A_{1,R}\big]\,dx\\
&\quad-2\int_{\mathbb{R}^3_+}\mathbf{u}\cdot\big[2(\nabla\partial_i\phi_R)\times\partial_i v_R+\nabla\phi_R\times\Delta v_R+(\nabla\Delta\phi_R)\times\mathbf{v}_R\big]\,dx\\
&\leqslant\|\phi_R\nabla\mathbf{u}\|_2\|\nabla A_{1,R}\|_2+\big\|\phi_R^{\frac12}\mathbf{u}\big\|_2\|\nabla A_{1,R}\|_2+\|\mathbf{u}\|_{\frac65}\|A_{1,R}\|_6\\
&\quad+\|\mathbf{u}\|_2\|\nabla v_R\|_2+\big\|\phi_R^{\frac12}\mathbf{u}\big\|_2\|\Delta\mathbf{v}_R\|_2+\|\mathbf{u}\|_2\|\mathbf{v}_R\|_6.
\end{aligned}\tag{6}$$

Here we noted that $|\nabla\phi_R|\leqslant c\phi_R^{1/2}$, $|\nabla^2\phi_R|\leqslant c$, and $\nabla^3\phi_R$ is compactly supported in $D_R=\{x\colon\ |x|\leqslant 2R\}$ with $|\nabla^3\phi_R|\leqslant\frac{c}{R}$. Applying the estimates for $\mathbf{v}_R$, $\mathbf{A}_{1,R}$, in Lemmas 2.1, 2.2 to the right-hand side of (6), we have

$$III \leqslant \|\phi_R \nabla \mathbf{u}\|_2 \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2 + \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2^2 + \|\mathbf{u}\|_{\frac{6}{5}} \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2 + \|\mathbf{u}\|_2 \|\nabla v_R\|_2. \tag{7}$$

By Lemma 2.3 we note that

$$\big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2^2 \leqslant Y_R^{\frac{1}{2}} \|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_{\frac{6}{5}}^2,$$

hence the right-hand side of (7) can be rewritten by

$$III \leqslant Y_R^{\frac{3}{4}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + Y_R^{\frac{1}{2}} \|\mathbf{u}\|_{\frac{6}{5}} + Y_R^{\frac{1}{4}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}^2 + \|\mathbf{u}\|_2 X^{\frac{1}{2}}. \tag{8}$$

We now estimate IV:

$$\begin{aligned} IV &= 2\int_{\mathbb{R}^3_+} \mathbf{u}\cdot U_i\big[(\partial_i\phi_R)\mathbf{A}_{1,R} + \phi_R \partial_i \mathbf{A}_{1,R} + (\nabla\partial_i\phi_R)\times \mathbf{v}_R + (\nabla\phi_R)\times\partial_i \mathbf{v}_R\big]\,dx \\ &\leqslant \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_3 \|\mathbf{U}\|_2 \|A_{1,R}\|_6 + \|\phi_R\mathbf{u}\|_6 \|\mathbf{U}\|_3 \|\nabla A_{1,R}\|_2 + \|\mathbf{u}\|_3 \|\mathbf{U}\|_2 \|\mathbf{v}_R\|_6 + \|\mathbf{u}\|_3 \|\mathbf{U}\|_6 \|\nabla \mathbf{v}_R\|_2. \end{aligned} \tag{9}$$

Note that

$$\begin{aligned} &\big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_3 \leqslant \|\phi_R\mathbf{u}\|_6^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}}, \\ &\|\phi_R\mathbf{u}\|_6 \leqslant \big\|\nabla(\phi_R\mathbf{u})\big\|_2 \leqslant \|\phi_R\nabla\mathbf{u}\|_2 + \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2, \\ &\|\mathbf{u}\|_3 \leqslant \|\mathbf{u}\|_6^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \leqslant \|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}}. \end{aligned}$$

With the above estimate and Lemmas 2.1, 2.2, the right-hand side of (9) is bounded by

$$\begin{aligned} IV &\leqslant \big(\|\phi_R\nabla\mathbf{u}\|_2^{\frac{1}{2}} + \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2^{\frac{1}{2}}\big)\|\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{U}\|_2 \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2 + \big(\|\phi_R\nabla\mathbf{u}\|_2 + \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2\big)\|\nabla\mathbf{U}\|_2^{\frac{1}{2}} \|\mathbf{U}\|_2^{\frac{1}{2}} \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2 \\ &\quad + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{U}\|_2 \|\nabla\mathbf{v}_R\|_2 + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\nabla\mathbf{U}\|_2 \|\nabla\mathbf{v}_R\|_2. \end{aligned} \tag{10}$$

We recall

$$\big\|\nabla^k\mathbf{U}\big\|_q \leqslant \big\|\nabla^k\mathbf{u}\big\|_2, \quad k = 0, 1.$$

Then the right-hand side of (10) can be rewritten by

$$\begin{aligned} IV &\leqslant \big(\|\phi_R\nabla\mathbf{u}\|_2^{\frac{1}{2}} + \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2^{\frac{1}{2}}\big)\|\mathbf{u}\|_2^{\frac{3}{2}} \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2 + \big(\|\phi_R\nabla\mathbf{u}\|_2 + \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2\big)\|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2 \\ &\quad + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{3}{2}} \|\nabla\mathbf{v}_R\|_2 + \|\nabla\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\nabla\mathbf{v}_R\|_2. \end{aligned} \tag{11}$$

Again recall

$$\big\|\phi_R^{\frac{1}{2}}\mathbf{u}\big\|_2^2 \leqslant Y_R^{\frac{1}{2}} \|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_{\frac{6}{5}}^2,$$

then we have

$$\begin{aligned} IV &\leqslant \big(Y_R^{\frac{1}{4}} + \big(Y_R^{\frac{1}{8}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{4}} + \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}}\big)\big)\|\mathbf{u}\|_2^{\frac{3}{2}} \big(Y_R^{\frac{1}{4}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}\big) \\ &\quad + \big(Y_R^{\frac{1}{2}} + \big(Y_R^{\frac{1}{4}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}\big)\big)\|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \big(Y_R^{\frac{1}{4}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}\big) + \big(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}}\big)X_R^{\frac{1}{2}} \\ &\leqslant \big(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}}\big)X_R^{\frac{1}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} Y_R^{\frac{3}{4}} + \big(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}}\big)Y_R^{\frac{1}{2}} \\ &\quad + \|\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{4}} Y_R^{\frac{3}{8}} + \big(\|\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_{\frac{6}{5}} + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}}\big)Y_R^{\frac{1}{4}} + \|\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{5}{4}} Y_R^{\frac{1}{8}} \\ &\quad + \|\mathbf{u}\|_2^{\frac{3}{2}} \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \|\mathbf{u}\|_{\frac{6}{5}}^2. \end{aligned} \tag{12}$$

Likewise I and II are bounded by

$$I \leqslant c\left\|\phi_R^{\frac{1}{2}}\mathbf{u}\right\|_2^2 \leqslant Y_R^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_{\frac{6}{5}}^2, \tag{13}$$

$$\begin{aligned} II &\leqslant c\left\|\phi_R^{\frac{1}{2}}\mathbf{u}\right\|_3^3 \leqslant c\|\phi_R\mathbf{u}\|_6^{\frac{3}{2}}\|\mathbf{u}\|_2^{\frac{3}{2}} \leqslant c\left\|\nabla(\phi_R\mathbf{u})\right\|_2^{\frac{3}{2}}\|\mathbf{u}\|_2^{\frac{3}{2}} \\ &\leqslant c\|\phi_R\nabla\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_2^{\frac{3}{2}} + c\left\|\phi_R^{\frac{1}{2}}\mathbf{u}\right\|_2^{\frac{3}{2}}\|\mathbf{u}\|_2^{\frac{3}{2}} \\ &\leqslant cY_R^{\frac{3}{4}}\|\mathbf{u}\|_2^{\frac{3}{2}} + c\|\mathbf{u}\|_2^{\frac{3}{2}}\left(Y_R^{\frac{3}{8}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{4}} + \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}}\right) \\ &\leqslant cY_R^{\frac{3}{4}}\|\mathbf{u}\|_2^{\frac{3}{2}} + \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{4}}Y_R^{\frac{3}{8}} + c\|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}}. \end{aligned} \tag{14}$$

Therefore our inequality is reduced to

$$\begin{aligned} \frac{d}{dt}X_R(t) + 2Y_R(t) \leqslant{}& \left(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}} + \|\mathbf{u}\|_2\right)X_R^{\frac{1}{2}} + \left(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + \|\mathbf{u}\|_2^{\frac{3}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}}\right)Y_R^{\frac{3}{4}} \\ &+ \left(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}\right)Y_R^{\frac{1}{2}} + \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{4}}Y_R^{\frac{3}{8}} \\ &+ \left(\|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}} + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}}\right)Y_R^{\frac{1}{4}} + \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{5}{4}}Y_R^{\frac{1}{8}} \\ &+ \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^2 + c\|\mathbf{u}\|_{\frac{6}{5}}^2. \end{aligned} \tag{15}$$

By Young's inequality we have

$$\begin{aligned} &\left(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + \|\mathbf{u}\|_2^{\frac{3}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}}\right)Y_R^{\frac{3}{4}} \leqslant \epsilon Y_R(t) + c_\epsilon\left(\|\nabla\mathbf{u}\|_2^2\|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^2 + \|\mathbf{u}\|_2^6 + \|\mathbf{u}\|_{\frac{6}{5}}^2\right), \\ &\left(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{1}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}\right)Y_R^{\frac{1}{2}} \leqslant \epsilon Y_R(t) + c_\epsilon\left(\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\mathbf{u}\|_{\frac{6}{5}}^2 + \|\mathbf{u}\|_2^3\|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_{\frac{6}{5}}^2\right), \\ &\|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{4}}Y^{\frac{3}{8}} \leqslant \epsilon Y_R(t) + c_\epsilon\|\mathbf{u}\|_2^{\frac{12}{5}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{6}{5}}, \\ &\left(\|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}} + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}} + \|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}}\right)Y_R^{\frac{1}{4}} \leqslant \epsilon Y_R(t) + c_\epsilon\left(\|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{4}{3}} + \|\nabla\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{6}{5}}^2 + \|\mathbf{u}\|_{\frac{6}{5}}^2\right), \\ &\|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{5}{4}}Y_R^{\frac{1}{8}} \leqslant \epsilon Y_R(t) + c_\epsilon\|\mathbf{u}\|_2^{\frac{12}{7}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{10}{7}}. \end{aligned}$$

Taking ϵ small enough, the above Gronwall inequality (15) reduces to

$$\begin{aligned} \frac{d}{dt}X_R(t) + 2Y_R(t) \leqslant{}& \left(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}} + \|\mathbf{u}\|_2\right)X_R^{\frac{1}{2}} + \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^2 + c\|\mathbf{u}\|_{\frac{6}{5}}^2 \\ &+ \|\mathbf{u}\|_2^{\frac{12}{7}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{10}{7}} + \|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{4}{3}} + \|\nabla\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{6}{5}}^2 + \|\mathbf{u}\|_2^{\frac{12}{5}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{6}{5}} + c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\mathbf{u}\|_{\frac{6}{5}}^2 \\ &+ \|\mathbf{u}\|_2^3\|\mathbf{u}\|_{\frac{6}{5}} + \|\mathbf{u}\|_{\frac{6}{5}}^2 + c\|\nabla\mathbf{u}\|_2^2\|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^2 + \|\mathbf{u}\|_2^6. \end{aligned} \tag{16}$$

Now we apply Young's inequality to the first therm of the right-hand side of (16), then we have

$$\begin{aligned} &\left(\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{3}{2}} + \|\nabla\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}} + \|\mathbf{u}\|_2\right)X_R^{\frac{1}{2}} \\ &\quad\leqslant \left(\|\nabla\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2 + \|\mathbf{u}\|_2\right)X_R + \|\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2 + \|\mathbf{u}\|_2. \end{aligned}$$

Hence (16) can be reduced to

$$\begin{aligned} \frac{d}{dt}X_R(t) + 2Y_R(t) \leqslant{}& \left(\|\nabla\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2 + \|\mathbf{u}\|_2\right)X_R + \|\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2 + \|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}} \\ &+ \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^2 + c\|\mathbf{u}\|_{\frac{6}{5}}^2 + \|\mathbf{u}\|_2^{\frac{12}{7}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{10}{7}} + \|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{4}{3}} + \|\nabla\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{6}{5}}^2 \end{aligned}$$

$$+\|\mathbf{u}\|_2^{\frac{12}{5}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{6}{5}}+c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\mathbf{u}\|_{\frac{6}{5}}^2+\|\mathbf{u}\|_2^3\|\mathbf{u}\|_{\frac{6}{5}}+\|\mathbf{u}\|_{\frac{6}{5}}^2$$
$$+c\|\nabla\mathbf{u}\|_2^2\|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^2+\|\mathbf{u}\|_2^6+\|\mathbf{u}\|_2. \tag{17}$$

Lemma 3.2. *Let $A(t), B(t)\in L^1(0,\infty)$, and suppose $X_R(t)$ satisfies the following Gronwall inequality*:

$$X'(t)\leqslant A(t)+B(t)X(t),\quad t>0.$$

Suppose that $X(t)<\infty$ for each $t\geqslant 0$. Then there is C independent of t such that

$$X(t)\leqslant CX(0)+C,\quad t>0.$$

The proof of the above lemma is trivial, so we omit its proof.
Now we set

$$\begin{aligned}A(t)={}&\|\mathbf{u}\|_2^2+\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2+\|\mathbf{u}\|_2^{\frac{3}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{3}{2}}+\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_2^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{6}{5}}^2+c\|\mathbf{u}\|_{\frac{6}{5}}^2\\&+\|\mathbf{u}\|_2^{\frac{12}{7}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{10}{7}}+\|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{4}{3}}+\|\nabla\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{6}{5}}^2+\|\mathbf{u}\|_2^{\frac{12}{5}}\|\mathbf{u}\|_{\frac{6}{5}}^{\frac{6}{5}}\\&+c\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2\|\mathbf{u}\|_{\frac{6}{5}}^2+\|\mathbf{u}\|_2^3\|\mathbf{u}\|_{\frac{6}{5}}+\|\mathbf{u}\|_{\frac{6}{5}}^2+c\|\nabla\mathbf{u}\|_2^2\|\mathbf{u}\|_2^2\|\mathbf{u}\|_{\frac{6}{5}}^2+\|\mathbf{u}\|_2^6+\|\mathbf{u}\|_2\end{aligned}$$

and

$$B(t)=\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_2+\|\mathbf{u}\|_2.$$

Recall that a weak solution and so its approximating solution of the Navier–Stokes equation has the following decay rate (see Fujigaki and Miyakawa [8]):

$$\|\mathbf{u}(t)\|_p\leqslant c(1+t)^{-2+3/(2p)},\quad p\in(1,2],$$

if $\mathbf{u}_0\in L^1\cap L^2$, $x_3\mathbf{u}_0\in L^1$. Thus we have $A(t)\in L^1(0,\infty)$, $B(t)\in L^1(0,\infty)$. Apply the above lemma to obtain

$$X_R(t)\leqslant CX_R(0)+C$$

for some C independent of t and R, that is,

$$\|\nabla\mathbf{v}_R(t)\|_2\leqslant C\|\nabla\mathbf{v}_R(0)\|_2+C,\quad t>0.$$

By Lemmas 2.1 and 2.2

$$\|\nabla\mathbf{v}_R(0)\|_2\leqslant c\|\phi_R\mathbf{u}_0\|_2+c\|\phi_R^{\frac{1}{2}}\mathbf{u}_0\|_{\frac{6}{5}}\leqslant c\|x_3^2\mathbf{u}_0\|_2+c\|x_3\mathbf{u}_0\|_{\frac{6}{5}}.$$

Hence we have that

$$\|\nabla\mathbf{v}_R(0)\|_2\leqslant C\quad\text{for some } C \text{ independent of } R, \mathbf{u},$$

if $x_3^2\mathbf{u}_0\in L^2,\ x_3\mathbf{u}_0\in L^{\frac{6}{5}}$, and thus we have

$$\|\nabla\mathbf{v}_R(t)\|_2\leqslant C\quad\text{for some } C \text{ independent of } t, R, \mathbf{u},$$

if $x_3^2\mathbf{u}_0\in L^2,\ x_3\mathbf{u}_0\in L^{\frac{6}{5}}$.
Now the boundedness of $\|\phi_R^{1/2}\mathbf{u}\|_2$ comes from the following lemma.

Lemma 3.3. *If $\mathbf{v}_R=N*(\phi_R\operatorname{curl}\mathbf{u})$, then*

$$\|\phi_R^{\frac{1}{2}}\mathbf{u}\|_2^2\leqslant cX_R^{\frac{1}{2}}(t)\|\mathbf{u}\|_2+c\|\mathbf{u}\|_{\frac{6}{5}}^2.$$

Proof. Since

$$\phi_R \mathbf{u} = \operatorname{curl} \mathbf{v}_R - \mathbf{A}_{1,R},$$

we have

$$\|\phi_R^{\frac{1}{2}} \mathbf{u}\|_2^2 = \langle \phi_R \mathbf{u}, \mathbf{u} \rangle = \langle \operatorname{curl} \mathbf{v}_R - \mathbf{A}_{1,R}, \mathbf{u} \rangle.$$

Hence by Sobolev inequality and Young's inequality we have

$$\begin{aligned}
\|\phi_R^{\frac{1}{2}} \mathbf{u}\|_2^2 &\leqslant c\|\nabla \mathbf{v}_R\|_2 \|\mathbf{u}\|_2 + \|\mathbf{A}_{1,R}\|_6 \|\mathbf{u}\|_{\frac{6}{5}} \\
&\leqslant c\|\nabla \mathbf{v}_R\|_2 \|\mathbf{u}\|_2 + c\|\phi_R^{\frac{1}{2}} \mathbf{u}\|_{L^2} \|\mathbf{u}\|_{\frac{6}{5}} \\
&\leqslant c\|\nabla \mathbf{v}_R\|_2 \|\mathbf{u}\|_2 + \epsilon\|\phi_R^{\frac{1}{2}} \mathbf{u}\|_{L^2}^2 + c_\epsilon \|\mathbf{u}\|_{\frac{6}{5}}^2
\end{aligned}$$

for some constant C_ϵ depending on ϵ. Taking $\epsilon \ll 1$ we have

$$\|\phi_R^{\frac{1}{2}} \mathbf{u}\|_2^2 \leqslant c\|\nabla \mathbf{v}_R\|_2 \|\mathbf{u}\|_2 + c\|\mathbf{u}\|_{\frac{6}{5}}^2. \qquad \square$$

Moreover, applying $\|\mathbf{u}(t)\|_2 \leqslant c(1+t)^{-5/4}$ and $\|\mathbf{u}(t)\|_{\frac{6}{5}} \leqslant c(1+t)^{-3/4}$ to the above lemma, we conclude that

$$\|\phi_R^{\frac{1}{2}} \mathbf{u}(t)\|_2 \leqslant c(1+t)^{-\frac{5}{8}}.$$

By sending R to ∞, we obtain

$$\|x_3 \mathbf{u}(t)\|_{L^2(\mathbf{R}_3^+)} \leqslant c(1+t)^{-\frac{5}{8}}.$$

This leads to the completeness of the proof of Theorem 3.1.

Appendix A. Approximating solutions

We consider the approximate solutions $\mathbf{u}^n$, $n = 1, 2, \ldots,$ of (1) with initial data $\mathbf{u}_0 \in L^1 \cap L^2$, $\operatorname{div} \mathbf{u}_0 = \underline{0}$, of the following equations in the domain Ω:

$$\begin{aligned}
&\frac{\partial}{\partial t} \mathbf{u}^n - \Delta \mathbf{u}^n + (\mathbf{U}^n \cdot \nabla) \mathbf{u}^n + \nabla p^n = 0, \quad t > 0, \\
&\nabla \cdot \mathbf{u}^n = 0, \\
&\mathbf{u}^n(0) = \mathbf{U}_0^n,
\end{aligned} \tag{A.1}$$

with boundary condition $\mathbf{u}^n = 0$ on $\partial\Omega$ and $\mathbf{u}^n = 0$ at infinity, where $\mathbf{U}^n$ and $\mathbf{U}_0^n$ are the mollification of $\mathbf{u}^n$ and $\mathbf{u}_0$, respectively, defined by

$$\mathbf{U}^n(x,t) = \int J_{1/n}(x-y) \mathbf{u}^n(x-y)\, dy,$$

and

$$\mathbf{U}_0^n(x,t) = \int J_{1/n}(x-y) \mathbf{u}_0(x-y)\, dy.$$

Here $J_{1/n} = n^3 J(n(x-y))$ for some mollifier J. One easily verifies that

$$\|\nabla^k \mathbf{U}^n(t)\|_p \leqslant \|\nabla^k \mathbf{u}^n(t)\|_p, \quad 1 \leqslant p \leqslant \infty,\ k = 0, 1. \tag{A.2}$$

(This kind of approximation was originally introduced by J. Leray [19] and also considered by [9].)

It may be shown that our weak solution $\mathbf{u}$ satisfies the properties as in [9], such as "generalized energy inequality" in the sense of Cafarelli–Kohn–Nirenberg, measurable pressure term, etc., when the initial velocity has some differentiability as in [9].

The solution $\mathbf{u}^n$ has the following properties:

Lemma A.1.

(a) $\mathbf{u}^n$ *exist uniquely in* $L^2(0,\infty; W^{1,2}(\Omega)) \cap L^\infty(0,\infty; L^2_\sigma(\Omega))$. *Moreover,*

$$\left\|\mathbf{u}^n(t)\right\|_{L^2}^2 + 2\int_0^t \left\|\nabla \mathbf{u}^n(\tau)\right\|_{L^2}^2 d\tau = \|\mathbf{u}_0\|_{L^2}^2, \quad t > 0.$$

(b) *There is a subsequence of* $\mathbf{u}^n$ *which converges in* $L^2_{\mathrm{loc}}(\Omega \times [0,\infty))$ *to a weak solution* $\mathbf{u}$ *of the Navier–Stokes equations* (1). *Moreover, by lower semi-continuity,* $\mathbf{u}$ *satisfies*

$$\left\|\mathbf{u}(t)\right\|_2^2 + \int_0^t \left\|\nabla \mathbf{u}(s)\right\|_2^2 ds \leqslant \|\mathbf{u}_0\|_2^2, \quad \text{for all } t > 0.$$

(c) $\mathbf{U}^n$ *and their spatial derivatives are continuous and bounded on* $\mathbb{R}^3_+ \times [0,T]$, $T > 0$, *and satisfy* $\nabla \cdot \mathbf{U}^n = 0$ *and*

$$\left\|\mathbf{U}^n(t)\right\|_{L^p} \leqslant \left\|\mathbf{u}^n(t)\right\|_{L^p}, \qquad \left\|\nabla \mathbf{U}^n(t)\right\|_{L^p} \leqslant \left\|\nabla \mathbf{u}^n(t)\right\|_{L^p}, \quad \text{for all } t > 0.$$

(d) $\mathbf{u}^n$ *has the following temporal decay rate*:

$$\left\|\mathbf{u}^n(t)\right\|_p \leqslant c(1+t)^{-2+3/(2p)}, \quad p \in (1,2],$$

if $\mathbf{u}_0 \in L^1 \cap L^2$, $x_3\mathbf{u}_0 \in L^1$.

Recall that a weak solution of the Navier–Stokes equation has the following decay rate (see Fujigaki and Miyakawa [8]):

$$\left\|\mathbf{u}(t)\right\|_p \leqslant c(1+t)^{-2+3/(2p)}, \quad p \in (1,2],$$

if $\mathbf{u}_0 \in L^1 \cap L^2$, $(1+x_3)\mathbf{u}_0 \in L^1$. By the same argument as in Fujigaki and Miyakawa [8], it is not hard to get the estimate in the above item (d).

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